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NOTE ON THE PERIODIC POINTS OF THE BILLIARD

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Marek Rychlik [2, Theorem 1.1] proves that for any bounded convex domain Ω in a Euclidean plane $\mathbf{R}^2$ with smooth *boundary* $X = \partial\Omega$ the set Fix_3 of all periodic points of period 3 of the billiard ball map related to Ω has empty interior in its (two-dimensional) phase space M_Ω. The last part of the proof of this theorem, considered in [2], involves a symbolic computation system. In this note a short elementary argument is presented which completes the proof in [2] without use of any computer programs. Combining this argument with §3 in [2], one gets also a direct proof of Theorem 1.2 of [2]: Fix_3 has Lebesgue measure zero.

We use the notation from [2], and state the results of [2] in a little more general form.

Theorem. *Let Ω be a bounded (note necessarily convex) domain in $\mathbf{R}^2$ with C^3-smooth boundary X. Then* Fix_3 *has empty interior and Lebesgue measure zero in M_Ω.*

Proof. Let $y_1, \cdots, y_n$ be the successive (transversal) reflection points of a periodic billiard trajectory in Ω. Consider a natural parametrization $h_i(x_i)$, $x_i \in \mathbf{R}$, of X around y_i with $\|h_i'(x_i)\| \equiv 1$, $\cos\varphi_i = \langle e_i, \nu_i\rangle > 0$, where $\nu_i = \nu(x_i)$ is the *unit normal* to X at $h_i(x_i)$, pointing into Ω, $\langle\cdot,\cdot\rangle$ is the natural inner product in $\mathbf{R}^2$, φ_i is the angle between e_i and ν_i, $0 < \varphi_i < \pi/2$, and

$$e_i = \frac{h_{i+1}(x_{i+1}) - h_i(x_i)}{\|h_{i+1}(x_{i+1}) - h_i(x_i)\|}.$$

One can introduce Φ_i and $\widehat{\Phi}_i$ simply by setting $\Phi_i = \cos\varphi_i$ and $\widehat{\Phi}_i = \sin\varphi_i$. Then, if $h_i(x_i)$ are the reflection points of a periodic trajectory, a simple computation gives

$$\frac{\partial l(x_i, x_{i+1})}{\partial x_i} = -\langle e_i, h_i'\rangle = -\cos\varphi_i = -\Phi_i = -\frac{\partial l(x_{i-1}, x_i)}{\partial x_i}.$$

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Set $k_i = k_i(x_i) = \langle h_i''(x_i), \nu_i \rangle$; then $h_i''(x_i) = k_i \nu_i$ (k_i is in fact the curvature function along $h_i(x_i)$). Now we have

$$\frac{\partial^2 l(x_i, x_{i+1})}{\partial x_i^2} = -\langle e_i, h_i'' \rangle + \frac{1}{l(x_i, x_{i+1})} - \frac{\cos^2 \varphi_i}{l(x_i, x_{i+1})}$$
$$= \frac{\sin^2 \varphi_i}{l(x_i, x_{i+1})} - k_i \sin \varphi_i.$$

In a similar way one finds

$$\frac{\partial^2 l(x_i, x_{i+1})}{\partial x_i \partial x_{i+1}} = \frac{\sin \varphi_i \sin \varphi_{i+1}}{l(x_i, x_{i+1})},$$

which implies that the matrix $d^2 \mathscr{L}_n(x)$ has the form (2.11), with (2.12) of [2]; these formulas are not new; cf., for example, [1] or [3].

Now suppose that Fix_3 contains a nonempty open subset of the phase space M_Ω. We may assume that $y = (y_1, y_2, y_3) \in U \subset \mathrm{Fix}_3$ for some open connected subset U of M_Ω. Since U consists of critical points of $\mathscr{L}_3$, we have

$$\mathscr{L}_3(x) = l_1 + l_2 + l_3 = c = \text{const} \tag{1}$$

for all $x = (x_1, x_2, x_3)$ with $(h_1(x_1), h_2(x_2), h_3(x_3)) \in U$. As in Proposition 2.1 of [2] one gets

$$k_1 = \frac{(l_2 + l_3 - l_1) \sin \varphi_1}{2 l_2 l_3} = \frac{(c - 2l_1) \sin \varphi_1}{2 l_2 l_3}. \tag{2}$$

By the Cosyne theorem (cf. Lemma 2.4 in [2]), $\sin \varphi_1 = \frac{1}{2}\sqrt{c(c-2l_1)/(l_2 l_3)}$. Combining the latter with (2) we find

$$4k_1^2 = g(x_1, x_2, x_3) = \frac{c(c - 2l_1)^3}{l_2^3 l_3^3}, \tag{3}$$

which is (2.20) of [2] for $i = 1$. Applying Proposition 2.2 of [2] and differentiating (3) in direction $(0, -\widehat{\Phi}_3, \widehat{\Phi}_2) = (0, -\sin \varphi_3, \sin \varphi_2)$, one obtains

$$0 = \frac{\partial g}{\partial x_2}(x_1, x_2, x_3)(-\sin \varphi_3) + \frac{\partial g}{\partial x_3}(x_1, x_2, x_3)(\sin \varphi_2). \tag{4}$$

On the other hand, (3) implies

$$\begin{aligned}\frac{\partial g}{\partial x_2} &= \frac{3c(c-2l_1)^2}{l_2^3 l_3^3}\left(-2\frac{\partial l_1}{\partial x_2}\right) - \frac{3c(c-2l_1)^3}{l_2^3 l_3^4}\left(\frac{\partial l_3}{\partial x_2}\right) \\ &= \frac{3c(c-2l_1)^2 \sin\varphi_2}{l_2^3 l_3^4}[2l_3 - (c-2l_1)] \\ &= \frac{3c(c-2l_1)^2 (c-2l_2)\sin\varphi_2}{l_2^3 l_3^4} > 0, \\ \frac{\partial g}{\partial x_3} &= \frac{3c(c-2l_1)^2}{l_2^3 l_3^3}\left(-2\frac{\partial l_1}{\partial x_3}\right) - \frac{3c(c-2l_1)^3}{l_2^4 l_3^3}\left(\frac{\partial l_2}{\partial x_3}\right) \\ &= -\frac{3c(c-2l_1)^2 \sin\varphi_3}{l_2^4 l_3^3}[2l_2 - (c-2l_1)] \\ &= -\frac{3c(c-2l_1)^2 (c-2l_3)\sin\varphi_3}{l_2^4 l_3^3} < 0.\end{aligned}$$

Therefore, the right-hand side of (4) is strictly negative, which is a contradiction. This proves the first part of the theorem.

For the second part, suppose that Fix_3 has positive Lebesgue measure in M_Ω. Then (1) might be not true; however, assuming that y is a Lebesgue density point of Fix_3, we have $\mathscr{L}_3(x) = c + \mathscr{O}(\|x\|^2)$, $c = \text{const}$, for $(h_1(x_1), h_2(x_2), h_3(x_3)) \in \mathrm{Fix}_3$ close to y. Now using the argument from §3 of [2] and a modification of the above argument, one gets that (4) again holds.

References

[1] A. Katok & J.-M. Strelcyn, In collaboration with F. Ledrappier and F. Przyticki, *Smooth maps with singularities: Invariant manifolds, entropy and billiards*, Lecture Notes in Math., Vol. 1222, Springer, Berlin, 1986.

[2] M. Rychlik, *Periodic points of the billiard ball map in a convex domain*, J. Differential Geometry **30** (1989) 191–205.

[3] L. Stojanov, *An estimate from above of the number of periodic orbits for semi-dispersed billiards*, Comm. Math. Phys. **124** (1989) 217–227.

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